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Best Local Approximation in Several Variables

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I. INTRODUCTION

The notion of best local approximation of a function has been introduced and developed by Chui, Shisha, and Smith [2]. This notion may be briefly restated as follows. Let $f: R^n \to R$ be a function in a normed linear space X of functions with norm $\|\cdot\|$. Let V denote a subset of X. We wish to approximate f near a point $\mathbf{x}_0 \in R^n$ using an element of V. Let $\{Q_\delta\}_{\delta>0}$ denote a net of volumes containing $\mathbf{x}_0$ with diameters shrinking to zero as $\delta \to 0$. For each $\delta > 0$ we "find" a $v_\delta \in V$ which minimizes $\|(f-v)\chi_{Q_\delta}\|$, where χ_{Q_δ} is the characteristic function of Q_δ. If v_δ converges as $\delta \to 0$ to an element v^* of V then v^* is said to be the best local approximant of f at x_0 with respect to the net Q_δ.

The basic underlying results in R^1 were developed in [6] and [1, 2]. These results concerned first, the best local approximant of $f \in C^{m+1}[0, 1]$ from an $(m+1)$ dimensional subspace of $C^{m+1}[0, 1]$. It was found that in the

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uniform norm the best local approximant of f at $x_0 = 0$ is obtained by choosing the element out of the subspace matching the first m derivatives of f at 0 (an assumption leading to the existence and uniqueness of this element having been previously made). The so-called best local quasi-rational approximant of a function f was also considered. Here $f \in C^{m+n-1}[0, 1]$ is approximated by the ratio of functions v/w, where $v \in V$ with $V \subset C^{m+n-1}[0, 1]$ is an m dimensional subspace of functions and $w \in W_1$ with $W_1 \subset C^{m+n+1}[0, 1]$ is an $(n-1)$ dimensional affine space in which all functions satisfy $w(0) = 1$. In best local quasi-rational approximation a net of functions v_δ / w_δ is generated using the pair v_δ, w_δ which for each δ minimizes $\|(wf - v)\chi_{Q_\delta}\|$. The results obtained in one dimension using the uniform norm show that under suitable conditions on f, V, and W_1, the best local quasi-rational approximant of f at zero is obtained by choosing v^* and w^* so that the appropriate number of derivatives of $w^*f - v^*$ are zero at $x = 0$. It was observed in [2] that this procedure for generating best local quasi-rational approximants is the same as that used to generate the Padé approximants of a function in a neighborhood of zero.

In this paper we investigate best local approximation from finite dimensional subspaces and best local quasi-rational approximation of real valued functions of several variables using the L_p norm $1 \leqslant p \leqslant \infty$. As in previous work, our functions will be endowed with a suitable number of derivatives, and our approximating spaces will be assumed to have the ability to match a suitable number of derivatives of a function at zero. In an n dimensional domain, however, the shape of the members of the net of volumes $\{Q_\delta\}$ sometimes affects the limit of the net $\{v_\delta\}$ of approximants in that, depending on the choice of $\{Q_\delta\}$, the limit may not exist, or may exist but depend on the particular choice. We will therefore restrict the usage of "best local approximant" to apply only to those limit functions which are insensitive to the choice of Q_δ in the sense that the same limit arises for any net of volumes Q_δ which for each δ can be inscribed and circumscribed by cubes whose dimensions are of the same order of magnitude in δ.

Our results on best local approximation by subspaces essentially say that if there is a unique element of the subspace which matches the derivatives at zero of any given function in C^{m+1} up through all derivatives of order m, then the best local approximant of an $f \in C^{m+1}$ in the L_p norm at $\mathbf{x}_0 = \mathbf{0}$ exists and is equal to the element of the subspace which matches the derivatives up through order m of f. If the subspace lacks the condition of being able to match all derivatives, then the best local approximant will in general not exist; this is illustrated by example. Finally, we consider the best local quasi-rational approximant of a function f by a ratio v/w of functions. Here we show that if all derivatives up through order m of $wf - v$ can be uniquely put equal to zero at $\mathbf{x} = \mathbf{0}$ by appropriately choosing $v \in V$ and $w \in W_1$ with V and W_1 as described above, then for those choices, v/w is the best local quasi-rational approximant of f at $\mathbf{x}=\mathbf{0}$. This result is related to current attempts to develop a Padé approximation theory for functions of several variables.

II. Notation and Definitions

The notation and results related to analysis in R^n that are used in this paper may be found in Schumaker [5, pp. 502–508]. In particular $\mathbf{x}=(x_1, x_2, ..., x_n)$ is a generic point in R^n, $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, ..., \alpha_n)$ is an n-dimensional vector (or multi-index) of nonnegative integers, $D^{\boldsymbol{\alpha}}$ denotes the mixed partial differentiation operator $D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n}$ in R^n, and $|\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_n$ is then the order of the derivative. The space of functions having continuous derivatives up through order m and defined on the open domain Ω is denoted $C^m(\Omega)$. We will be using the norm $\|f\|_{L_p(Q)}$ which is $[\int_Q |f(x)|^p\, dx]^{1/p}$ if $1 \leqslant p < \infty$, and $\sup_Q |f|$ if $p=\infty$, where f is a measurable function and Q is a measurable subset of Ω. We will make considerable use of Taylor's theorem: if $\mathbf{x}$ and $\mathbf{y}$ and the line segment from $\mathbf{y}$ to $\mathbf{x}$ lie in Ω and $f \in C^{m+1}(\Omega)$ then $f(x)=\sum_{|\boldsymbol{\alpha}| \leqslant m} (D^{\boldsymbol{\alpha}} f(\mathbf{y})(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}/\boldsymbol{\alpha}!) + O(\sum_{|\boldsymbol{\alpha}|=m+1} |\mathbf{x}-\mathbf{y}|^{\boldsymbol{\alpha}})$, where $(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}=(x_1-y_1)^{\alpha_1} \cdots (x_n-y_n)^{\alpha_n}$ and $\boldsymbol{\alpha}!=\alpha_1!\,\alpha_2! \cdots \alpha_n!$.

Throughout this paper $\{Q_\delta\}_{\delta>0}$ will denote a net of measurable subsets of R^n (referred to here as volumes) which contain the origin $\mathbf{0}$ and which have the property that for each δ, Q_δ contains a cube of side r_δ and is contained in a cube of side δ with δ/r_δ bounded as a function of δ for all $\delta>0$.

Let f be a real valued function defined and continuous on some bounded open subset Ω of R^n which contains $\mathbf{0}$ and let $\{Q_\delta\}_{\delta>0}$ denote a net of volumes as described above. Let V be a finite dimensional subspace of functions continuous on Ω. For each δ, let $v_\delta \in V$ satisfy $\|f-v_\delta\|_{L_p(Q_\delta)}=\min_{v\in V} \|f-v\|_{L_p(Q_\delta)}$. If as $\delta \to 0$, $v_\delta \to v^*$ (convergence being in $L_p(\Omega)$) and the limit v^* is independent of the choice of $\{Q_\delta\}$ then we call v^* *the best local approximant of f at* $\mathbf{0}$ from the space V with respect to the L_p norm.

Let f be as above and let V and W be finite dimensional subspaces of functions continuous on Ω. Denote by W_1 the affine space $\{w \in W: w(\mathbf{0})=1\}$. Let $\{Q_\delta\}$ be a net of volumes. For each δ, let w_δ and v_δ satisfy $\|w_\delta f-v_\delta\|_{L_p(Q_\delta)}=\min_{w\in W_1, v\in V} \|wf-v\|_{L_p(Q_\delta)}$. If as $\delta \to 0$, $v_\delta/w_\delta \to v^*/w^*$, and the limit is independent of the choice of $\{Q_\delta\}$, then we call v^*/w^* the *best local quasi-rational approximant* of f at $\mathbf{0}$ from V and W with respect to the L_p norm. (Note that this v^* has no particular relation with the v^* of best local approximation above. Also, we do not require separately that $v_\delta \to v^*$ and $w_\delta \to w^*$; however, in our theorem the assumptions will be such that these limits in fact do exist separately.)

A vector space V of functions in $C^m(\Omega)$, $\mathbf{0} \in \Omega$, is said to be *uniquely* *interpolating* at $\mathbf{0}$ of order m if there is a unique $v \in V$ with prescribed derivatives up through order m. If $u_i(\mathbf{x})$, $i = 1,..., k$, is a basis of V, then the condition of being uniquely interpolating at zero of order m is equivalent to the Wronskian determinant being nonzero. This is the determinant of the square matrix $(D^{\alpha}u_i(\mathbf{0}))$, $|\alpha| \leqslant m$, $i = 1,..., k$, where each value of α corresponds to a row of the matrix and k and m are related by $k = \text{card}\{\alpha: |\alpha| \leqslant m\}$.

III. Results

Our results in this section are mainly sufficient conditions for the existence of the best local approximant and the best local quasi-rational approximant for real valued functions on R^n. These results hold independent of the p in L_p, $1 \leqslant p \leqslant \infty$. Basically, Taylor's theorem gives a certain order of local approximation, if derivatives of an appropriately smooth function are matched by a function from the approximating set. Our results state that if from the approximating set, there is a unique function for which all derivatives up through order m of the error ($f - v$ in one case or $wf - v$ in the quasi-rational case) can be put to zero (m a nonnegative integer), then that function is the best local approximant; the net of best approximants (v_δ or v_δ / w_δ) generated by the net Q_δ of volumes converges to it on the domain Ω.

Proposition. *Suppose the subspace $V \subset C^{m+1}(\Omega)$ has basis $u_i(\mathbf{x})$, $i = 1,..., k$, and that V is uniquely interpolating at $\mathbf{0}$ of order m. If, as $\delta \to 0$ the functions $f_\delta(\mathbf{x}) = \sum_{i=1}^{k} a_{i,\delta} u_i(\mathbf{x})$ satisfy $\|f_\delta\|_{L_p(Q_\delta)} = o(\delta^{m+n/p})$, then for each i, $a_{i,\delta} \to 0$ as $\delta \to 0$ and $f_\delta(\mathbf{x}) \to 0$ in $L_p(\Omega)$.*

Proof. We will prove the result for the case of $|a_{i,\delta}|$ bounded uniformly in δ for each i. The general case will then follow from a simple argument.

We remark first that the $\|\cdot\|_{L_p(Q_\delta)}$ norm of a function of size $o(\sum_{|\alpha|=m} \mathbf{x}^{\alpha})$ is $o(\delta^{m+n/p})$ as we recall that Q_δ is contained in a cube of side δ. The Taylor expansion of $f_\delta(\mathbf{x})$ about $\mathbf{x} = \mathbf{0}$ up through order m is given by

$$f_\delta(\mathbf{x}) = \sum_{|\alpha| \leqslant m} D^{\alpha} f_\delta(\mathbf{0})\, \mathbf{x}^{\alpha}/\alpha! + o\left(\sum_{|\alpha|=m} \mathbf{x}^{\alpha}\right).$$

An important observation is that, as V is uniquely interpolating at $\mathbf{0}$, the $a_{i,\delta}$ are uniquely determined by the coefficients $D^{\alpha} f_\delta(\mathbf{0})$ via a matrix multiplication by the inverse of the matrix $D^{\alpha} u_i(\mathbf{0})$ which is independent of δ.

By hypothesis, $\|f_\delta\|_{L_p(Q_\delta)} = o(\delta^{m+n/p})$ from which it follows that

$$\left\| \sum_{|\alpha| \leqslant m} \frac{D^{\alpha} f_\delta(\mathbf{0})\, \mathbf{x}^{\alpha}}{\alpha!} \right\|_{L_p(Q_\delta)} = o(\delta^{m+n/p}). \tag{1}$$

We will show that $D\mathscr{Y}_\delta(\mathbf{0}) \to 0$ as $\delta \to 0$ and this in turn will imply that $a_{i,\delta} \to 0$.

It is not difficult to show that for a polynomial in n variables of the form $\sum_\alpha b_\alpha x^\alpha$ the simple scaling transformation $\mathbf{x} = \delta \mathbf{y}$ results in the equation

$$\left\| \sum_\alpha b_\alpha \mathbf{x}^\alpha \right\|_{L_p(Q_\delta)} = \delta^{n/p} \left\| \sum_\alpha \delta^{|\alpha|} b_\alpha \mathbf{y}^\alpha \right\|_{L_p(Q_\delta/\delta)},$$

where $Q_\delta/\delta = \{\mathbf{y} : \delta\mathbf{y} \in Q_\delta\}$. Applying this result to (1) and dividing both sides by $\delta^{m+n/p}$, we obtain

$$\left\| \sum_{|\alpha| \leqslant m} \frac{\delta^{|\alpha|-m} D^{\alpha}\mathscr{Y}_\delta(\mathbf{0})\, \mathbf{y}^\alpha}{\alpha!} \right\|_{L_p(Q_\delta/\delta)} = o(1). \tag{2}$$

Redefine $b_{\alpha,\delta} = (\delta^{|\alpha|-m} D^\alpha f_\delta(\mathbf{0})/\alpha!)$. We will prove that $b_{\alpha,\delta} \to 0$ by showing that every sequence $\{b_{\alpha,\delta_i}\}$ with $\delta_i \to 0$ has a subsequence $\{b_{\alpha,\delta_{i_j}}\}$ such that $b_{\alpha,\delta_{i_j}} \to 0$.

Recall that for each δ, Q_δ is contained within a cube of side δ and contains a cube, denoted by I_δ, which has sides at least as large as $c\delta$, with $c > 0$ a fixed constant. It follows then that Q_δ/δ is contained for all δ within the cube $\times_1^n [-1, 1]$ and that the cubes I_δ/δ have sides of length at least c. Now the centers of the cubes $\{I_{\delta_i}/\delta_i\}$ all lie within the cube $\times_1^n [-1, 1]$ so that there is a subsequence of $\{\delta_i\}$, $\{\delta_{i_k}\} \equiv \{\varepsilon_k\}$, such that the centers of the cubes $I_{\varepsilon_k}/\varepsilon_k$ converge. This in turn implies that for large k, the cubes $I_{\varepsilon_k}/\varepsilon_k$ have substantial intersection. Indeed, there exists a K such that $\bigcap_{k \geqslant K} (I_{\varepsilon_k}/\varepsilon_k)$ contains a cube B of side $c/4$ and then $B \subset Q_{\varepsilon_k}/\varepsilon_k$ for $k \geqslant K$. We then have

$$\lim_{k\to\infty} \left\| \sum_{|\alpha| \leqslant m} b_{\alpha,\varepsilon_k} y^\alpha \right\|_{L_p(B)} \leqslant \lim_{k\to\infty} \left\| \sum_{|\alpha| \leqslant m} b_{\alpha,\varepsilon_k} y^\alpha \right\|_{L_p(Q_{\varepsilon_k}/\varepsilon_k)} = 0,$$

where the last equality follows from (2). This implies finally that $b_{\alpha,\varepsilon_k} \to 0$ so that we have proved that $b_{\alpha,\delta} \to 0$ as $\delta \to 0$. From the definition of $b_{\alpha,\delta}$ we then obtain $D^{\alpha}\mathscr{Y}_\delta(\mathbf{0}) = o(\delta^{m-|\alpha|})$ as $\delta \to 0$, so that in particular $D^{\alpha}\mathscr{Y}_\delta(\mathbf{0}) \to 0$. This completes the proof in the case that the $a_{i,\delta}$, the components of $f_\delta(\mathbf{x})$ with respect to the basis $\{u_i(\mathbf{x})\}$ are uniformly bounded in δ. Suppose that the $a_{i,\delta}$ were not bounded in δ. Then we would have $\max_i |a_{i,\delta_j}| \to \infty$ for a sequence $\delta_j \to 0$. If we then define $g_{\delta_j}(\mathbf{x}) = [\max_i |a_{i,\delta_j}|]^{-1} f_{\delta_j}(\mathbf{x})$, then $\| g_{\delta_j} \|_{L_p(Q_{\delta_j})} = o(\delta_j^{m+n/p})$ and the components of g_{δ_j} with respect to the basis $\{u_i\}$ are uniformly bounded in j. The previous arguments then show that $g_{\delta_j} \to 0$ as $j \to \infty$. But one of the components of g_{δ_j} must have absolute value 1 for each j and a contradiction is obtained. This completes the proof of the proposition.

THEOREM 1. *Suppose the subspace $V \subset C^{m+1}(\Omega)$ is uniquely interpolating at* $\mathbf{0}$ *of order m. Then the best local approximant of $f \in C^{m+1}$ at* $\mathbf{0}$ *from V is the unique $v \in V$ whose derivatives up through order m match those of f at* $\mathbf{0}$.

Proof. By choosing $v^* \in V$ so as to match the derivatives up through order m of f at $\mathbf{0}$, we are guaranteed by Taylor's theorem that $\min_{v \in V} \|f - v\|_{L_p(Q_\delta)} \leqslant \|f - v^*\|_{L_p(Q_\delta)} = o(\delta^{m+n/p})$. If we write v_δ, the best L_p approximants of f on Q_δ, in the form $v_\delta = v^* + v_\delta^+$ we then have $o(\delta^{m+n/p}) \geqslant \|f - v_\delta\|_{L_p(Q_\delta)} = \|f - v^* - v_\delta^+\|_{L_p(Q_\delta)} \geqslant \|v_\delta^+\|_{L_p(Q_\delta)} - \|f - v^*\|_{L_p(Q_\delta)}$, so that $\|v_\delta^+\|_{L_p(Q_\delta)} = o(\delta^{m+n/p})$. But then the proposition implies that $\|v_\delta^+\| \to 0$ so that $v_\delta \to v^*$ and the theorem is proved.

Next we consider the best quasi-rational approximant of f. We let V and W denote finite dimensional subspaces of $C^{m+1}(\Omega)$, we set $W_0 = \{w \in W: w(\mathbf{0}) = 0\}$ and $W_1 = \{w \in W: w(\mathbf{0}) = 1\}$.

THEOREM 2. *Let V and W_0 be subspaces of functions in $C^{m+1}(\Omega)$ as defined above and let $f \in C^{m+1}(\Omega)$. Suppose the subspaces fW_0 and V are disjoint except for* $\mathbf{0}$. *If the vector space $\{fw - v: w \in W_0, v \in V\}$ is uniquely interpolating at* $\mathbf{0}$ *of order m, then the best local quasi-rational approximant of f at* $\mathbf{0}$ *is obtained by choosing the unique $v^* \in V$ and $w^* \in W_1$ so that all the derivatives of $fw^* - v^*$ up through order m are zero.*

Proof. First observe that if w_1 is a function in W with $w_1(\mathbf{0}) = 1$, then we can express W_1 as $W_1 = w_1 + W_0$. In that case $fw - v$ with $w \in W_1$ and $v \in V$ may be written as $fw_1 + (fw_0 - v)$ with $w_0 \in W_0$. The set of functions in parentheses was assumed uniquely interpolating up through order m so that there is indeed a unique $w_0^* \in W_0$ and $v^* \in V$ which makes $fw - v$ have all zero derivatives up through order m. (Nonuniqueness would imply the existence of a w_0 and v such that $fw_0 - v \equiv 0$, which contradicts the disjointness assumption in the statement of the theorem.) If $w_{0,\delta}$ and v_δ denote solutions of $\min_{w_0 \in W_0, v \in V} \|fw_1 + fw_0 - v\|_{L_p(Q_\delta)}$, then an argument similar to the one in Theorem 1 shows that $fw_{0,\delta} - v_\delta \to fw_0^* - v^*$. The proposition shows, moreover, that if we write out the components of $fw_{0,\delta} - v_\delta$ with respect to bases of fW_0 and V, then we may conclude separately that $w_{0,\delta} \to w_0^*$ and $v_\delta \to v^*$ and finally, that $w_\delta \equiv w_1 + w_{0,\delta} \to w^* \equiv w_1 + w_0^*$. This proves the theorem.

As a method of approximation of a function of several variables by polynomials in the quasi-rational sense, interpolation of derivatives arises as the central technique in the theory of Padé approximants. While the one variable theory is well developed, in several variables ambiguity arises in the choice of which derivatives should be matched when, as is usually the case, all derivatives up to order $m - 1$, say, can be matched but not all of those up to order m. See, for instance, Lutterodt [4] and Karllson and Wallin [3]. From the viewpoint of the best local approximant, however, it does not even make sense to consider a quasi-rational approximant unless the approximating space is uniquely interpolating of order m. This will be illustrated by an example.

Suppose we wish a best local approximant of the function xy using the vector space spanned by 1, x, y, x^2+xy+y^2. The vector space uniquely interpolates any specification of one second-order derivative and all derivatives up through the first. We will see, however, that the net of approximations generated by the volumes Q_δ depends sensitively on the dimensions of the volumes. Writing out explicitly the minimization problem, we have

$$\min \|f-v\|_{L_p(Q_\delta)} = \min \left\{ \int_0^{c\delta} \int_0^{\delta} |xy - a_0 - a_1 x - a_2 y - a_3(x^2+xy+y^2)|^p \, dx\, dy \right\}^{1/p},$$

where we are using for Q_δ a net of rectangles with dimensions $\delta \times c\delta$ $(c < 1)$. We now perform the scaling transformations $x = \delta s$, $y = c\,\delta t$, $a_0 = \delta^2 \gamma_0$, $a_1 = \delta\gamma_1$, $a_2 = \delta\gamma_2$, $a_3 = \gamma_3$, and obtain

$$\min \|f-v\|_{L_p(Q_\delta)} = \delta^{2+2/p} \min \left[\int_0^1 \int_0^1 |cst - \gamma_0 - \gamma_1 s - \gamma_2 ct - \gamma_3(s^2 + cst + c^2t^2)|^p \, ds\, dt \right]^{1/p}.$$

We see that the optimal choices of γ_0, γ_1, γ_2, and γ_3 are independent of δ, so that in particular, the optimal choice of a_3 is independent of δ. However, it clearly depends on the choice of the constant c, so that, although for each fixed c the approximations v_δ tend to a limit, this limit is not independent of c. This means, of course, that the limit is not the same for all nets Q_δ of rectangles and thus a best local approximant does not exist. This example clearly generalizes to more complicated situations; whenever the number of basis functions is such that interpolation up through order $m-1$ is possible, but not all the way up to order m, then the limit of the net of best approximants v_δ will depend on the dimensions of the net of volumes Q_δ (see *Note added in proof*).

Note added in proof. This dependence is treated in some detail in [7] as part of an analysis of multipoint local approximation.

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